PROOF OF THE COMBINATORIAL NULLSTELLENSATZ OVER INTEGRAL DOMAINS IN THE SPIRIT OF KOUBA

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Abstract. It is shown that by eliminating duality theory of vector spaces from a recent proof of Kouba (O. Kouba, A duality based proof of the Combinatorial Nullstellensatz. Electron. J. Combin. 16 (2009), #N9) one obtains a direct proof of the nonvanishing-version of Alon's Combinatorial Nullstellensatz for polynomials over an arbitrary integral domain. The proof relies on Cramer's rule and Vandermonde's determinant to explicitly describe a map used by Kouba in terms of cofactors of a certain matrix.

That the Combinatorial Nullstellensatz is true over integral domains is a well-known fact which is already contained in Alon's work and emphasized in recent articles of Michałek and Schauz; the sole purpose of the present note is to point out that not only is it not necessary to invoke duality of vector spaces, but by not doing so one easily obtains a more general result.

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1. Introduction

The Combinatorial Nullstellensatz is a very useful theorem (see [1]) about multivariate polynomials over an integral domain which bears some resemblance to the classical Nullstellensatz of Hilbert.

Theorem 1 (Alon, Combinatorial Nullstellensatz (ideal-containment-version), Theorem 1.1 in [1])**.** *Let K be a field, $R \subseteq K$ a subring, $f \in R[x_1, \ldots, x_n]$, $S_1, \ldots, S_n$ arbitrary nonempty subsets of K, and $g_i := \prod_{s \in S_i}(x_i - s)$ for every $1 \le i \le n$. If $f(s_1, \ldots, s_n) = 0$ for every $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$, then there exist polynomials $h_i \in R[x_1, \ldots, x_n]$ with the property that $\deg(h_i) \le \deg(f) - \deg(g_i)$ for every $1 \le i \le n$, and $f = \sum_{i=1}^n h_i g_i$.*

Theorem 2 (Alon, Combinatorial Nullstellensatz (nonvanishing-version), Theorem 1.2 in [1])**.** *Let K be a field, $R \subseteq K$ a subring, and $f \in R[x_1, \ldots, x_n]$. Let $c \cdot x_1^{d_1} \cdots x_n^{d_n}$ be a term in f with $c \neq 0$ whose degree $d_1 + \cdots + d_n$ is maximum among all degrees of terms in f. Then every product $S_1 \times \cdots \times S_n$, where each S_i is an arbitrary finite subset of R satisfying $|S_i| = d_i + 1$, contains at least one point $(s_1, \ldots, s_n)$ with $f(s_1, \ldots, s_n) \neq 0$.*

Three comments are in order. First, talking about subrings of a field is equivalent to talking about integral domains: every subring of a field clearly is an integral domain, and, conversely, every integral domain R is (isomorphic to) a subring of its field of fractions $\mathrm{Quot}(R)$. Second, strictly speaking, rings are mentioned in [1] only in Theorem 1, but Alon's proof in [1] of Theorem 2 is valid for polynomials

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over integral domains as well. Third, it is intended that the S_i are allowed to be subsets of K in Theorem 1 but required to be subsets of R in Theorem 2, since this is the slightly stronger formulation: if Theorem 2 is true as it is formulated here, then by invoking it with $R = K$ and by viewing an $f \in R[x_1, \dots, x_n]$, R being a subring of K, as a polynomial in $K[x_1, \dots, x_n]$, it is true as well with the S_i being allowed to be arbitrary subsets of K.

In [1], Theorem 2 was deduced from Theorem 1. In [3], Kouba gave a beautifully simple and direct proof of the nonvanishing-version of the Combinatorial Nullstellensatz, bypassing the use of the ideal-containment-version. Kouba's argument was restricted to the case of polynomials over a field and at one step applied a suitably chosen linear form on the vector space $K[x_1, \dots, x_n]$ to the given polynomial f in Theorem 2.

However, for Kouba's idea to work, it is not necessary to have recourse to duality theory of vector spaces and in the following section it will be shown how to make Kouba's idea work without it, thus obtaining a direct proof of the full Theorem 2.

Finally, two relevant recent articles ought to be mentioned. A very short direct proof of Theorem 2 was given by Michałek in [5] who explicitly remarks that the proof works for integral domains as well. Moreover, the differences $\{s - s' : \{s, s'\} \in \binom{S_k}{2}\}$ in the proof below play a similar role in Michałek's proof. In [6], Schauz obtained far-reaching generalizations and sharpenings of Theorem 2, expressly working with integral domains and generalizations thereof throughout the paper.

2. Proof of Theorem 2

The proof of the Theorem 2 will be based on the following simple lemma.

Lemma 3. *Let R be an integral domain. Let $S = \{s_1, \dots, s_m\} \subseteq R$ be an arbitrary finite subset. Then there exist elements $\lambda_1^{(S)}, \dots, \lambda_m^{(S)}$ of R such that*

$$\lambda_1^{(S)} \cdot (1, s_1, s_1^2, \dots, s_1^{m-1}) + \cdots + \lambda_m^{(S)} \cdot (1, s_m, s_m^2, \dots, s_m^{m-1}) \\ = (0, 0, 0, \dots, 0, \prod_{1 \le i < j \le m} (s_i - s_j)). \tag{1}$$

Proof. Let $[m] := \{1, \dots, m\}$. Define b to be the right-hand side of the claimed equation, taken as a column vector, and let $A = (a_{ij})_{(i,j) \in [m]^2}$ be the Vandermonde matrix defined by $a_{ij} := s_j^{i-1}$. Then the statement of the lemma is equivalent to the existence of a solution $\lambda^{(S)} \in R^m$ of the system of linear equations $A\lambda^{(S)} = b$. By the well-known formula for the determinant of a Vandermonde matrix (see [4], Ch. XIII, §4, example after Prop. 4.10), $\det(A) = \prod_{1 \le i < j \le m} (s_i - s_j)$.

Since S is a set, all factors of this product are nonzero, and since R has no zero divisors, the determinant is therefore nonzero as well. Now let α_{ij} be the cofactors of A, i.e. $\alpha_{ij} := (-1)^{i+j} \det(A^{(ij)})$, where $A^{(ij)}$ is the $(m-1) \times (m-1)$ matrix obtained from A by deleting the i-th row and the j-th column (see [2], Ch. IX, §3, before Lemma 1). By Cramer's rule (see Ch. IX, §3, Corollary 2 of Theorem 6 in [2] or Theorem 4.4 in [4]), for every $j \in [m]$,

$$\det(A) \cdot \lambda_j^{(S)} = \sum_{i=1}^{m} \alpha_{ij} b_i.$$

Using $b_m = \det(A)$, $b_i = 0$ for every $1 \le i < m$, and the commutativity of an integral domain, this reduces to

$$\det(A) \cdot \left(\lambda_j^{(S)} - \alpha_{mj}\right) = 0.$$

Hence, since $\det(A) \neq 0$ and R has no zero divisors, if follows that the cofactors $\lambda_j^{(S)} = \alpha_{mj} \in R$ provide explicit elements with the desired property. □

Using this lemma, Kouba's argument may now be carried out without change in the setting of integral domains.

Proof of Theorem 2. Let R be an arbitrary integral domain and $f \in R[x_1, \dots, x_n]$ be an arbitrary polynomial. Let $d_1, \dots, d_n \in \mathbb{N}_{\ge 0}$ be the exponents of a term $c \cdot x_1^{d_1} \cdots x_n^{d_n}$ with $c \neq 0$ which has maximum degree in f. For each $k \in [n]$, choose an arbitrary finite subset $S_k \subseteq R$ and apply Lemma 3 with $S = S_k$ and $m = |S| = d_k + 1$ to obtain a family of elements $(\lambda_{s_k}^{(S_k)})_{s_k \in S_k}$ of R (where in order to avoid double indices the coefficients λ are now being indexed by the elements of S_k directly, not by an enumeration of each S_k) with the property that

$$\sum_{s_k \in S_k} \lambda_{s_k}^{(S_k)} \cdot s_k^{\ell} = 0 \qquad \text{for every } \ell \in \{0, \dots, d_k - 1\}, \tag{2}$$

$$\sum_{s_k \in S_k} \lambda_{s_k}^{(S_k)} \cdot s_k^{d_k} = \prod_{\{s, s'\} \in \binom{S_k}{2}} (s - s') =: r_k \in R \backslash \{0\}. \tag{3}$$

Using the coefficient families $(\lambda_{s_k}^{(S_k)})_{s_k \in S_k}$, define, à la Kouba, the map

$$\begin{aligned} \Phi : \quad & R[x_1, \dots, x_n] \longrightarrow R \\ & g \longmapsto \sum_{(s_1, \dots, s_n) \in S_1 \times \cdots \times S_n} \lambda_{s_1}^{(S_1)} \cdots \lambda_{s_n}^{(S_n)} \cdot g(s_1, \dots, s_n). \end{aligned} \tag{4}$$

Due to the commutativity of an integral domain, Φ is an R-linear form on the R-module $R[x_1, \dots, x_n]$, hence its value $\Phi(f)$ on a polynomial f can be evaluated termwise as

$$\Phi(f) = \sum_{c \cdot t \text{ a term in } f} c \cdot \Phi(t). \tag{5}$$

If $t = c \cdot x_1^{d_1'} \cdots x_n^{d_n'}$ is an arbitrary term in $R[x_1, \dots, x_n]$, then

$$\begin{aligned} \Phi(t) = c \cdot \Phi(x_1^{d_1'} \cdots x_n^{d_n'}) &= c \cdot \sum_{(s_1, \dots, s_n) \in S_1 \times \cdots \times S_n} \lambda_{s_1}^{(S_1)} \cdots \lambda_{s_n}^{(S_n)} \cdot s_1^{d_1'} \cdots s_n^{d_n'} \\ &= c \cdot \sum_{s_1 \in S_1} \cdots \sum_{s_n \in S_n} \lambda_{s_1}^{(S_1)} \cdots \lambda_{s_n}^{(S_n)} \cdot s_1^{d_1'} \cdots s_n^{d_n'} \\ &= c \cdot \prod_{k=1}^{n} \left(\sum_{s_k \in S_k} \lambda_{s_k}^{(S_k)} s_k^{d_k'}, \right) \end{aligned} \tag{6}$$

where in the last step again use has been made of the commutativity of an integral domain. By (6) and (2) it follows that for every term t, if there is at least one exponent d_i' with $d_i' < d_i$, then $\Phi(t) = 0$. Moreover, by the choice of the term $c \cdot x_1^{d_1} \cdots x_n^{d_n}$, every term $c' \cdot x_1^{d_1'} \cdots x_n^{d_n'}$ of f which is different from the term $c \cdot$

$x_1^{d_1} \cdots x_n^{d_n}$ must, even if it has itself maximum degree in f, contain at least one exponent d_i' with $d_i' < d_i$. Therefore

$$\sum_{(s_1,\dots,s_n)\in S_1\times\cdots\times S_n} \lambda_{s_1}^{(S_1)}\cdots\lambda_{s_n}^{(S_n)}\cdot f(s_1,\dots,s_n) \overset{(4)}{=} \Phi(f) \overset{(2),(6)}{=} c\cdot\Phi(x_1^{d_1}\cdots x_n^{d_n}) =$$

$$\overset{(3),(6)}{=} c\cdot\prod_{k=1}^{n}\prod_{\{s,s'\}\in\binom{S_k}{2}}(s-s') = c\cdot\prod_{k=1}^{n} r_k \neq 0, \tag{7}$$

since R has no zero divisors. Obviously this implies that there exists at least one point $(s_1,\dots,s_n)\in S_1\times\cdots\times S_n$ where f does not vanish. $\square$

3. Concluding question

Is there any interesting use for the fact that even in the case of integral domains the coefficients of Kouba's map can be explicitly expressed in terms of cofactors of the matrices (s_j^{i-1})?

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